FORCING $\square_{\omega_1}$ WITH FINITE CONDITIONS

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Abstract. We show a construction of the square principle $\square_{\omega_1}$ by means of forcing with finite conditions.

1. Introduction

The square principle on a cardinal κ states that there is a sequence $\langle C_\alpha \rangle$ indexed by the limit ordinals in $[\kappa, \kappa^+)$ such that each C_α is a club subset of α of order type $\leq \kappa$ and the sequence is coherent in the sense that if β is a limit point of α then $C_\beta = C_\alpha \cap \beta$. This principle is a feature of the constructible universe $\mathbf{L}$ which was discovered by Jensen and used by him to show the existence of an ω_2-Souslin tree in $\mathbf{L}$ [7]. The related principle $\diamondsuit$, which was used to construct an ω_1-Souslin tree in $\mathbf{L}$ by Jensen, may also be added or destroyed by forcing as wished (see [10] for examples and discussion) and as is known of recently ([12]) at $\kappa \geq \omega_2$ which are successors of regular cardinals, it is simply equivalent to GCH. However, $\square$ is connected to large cardinals. For example, by an old proof of Solovay [13], square cannot hold above a supercompact cardinal, and on smaller cardinals, it cannot hold in the presence of forcing axioms, e.g. Todorčević [14] proved that PFA implies that for all $\kappa \geq \omega_2$, $\square_\kappa$ fails. Therefore $\square$ can be seen as a reflection principle inimical to large cardinals, and in fact by varying the definition of square by allowing a cardinal parameter which measures how many guesses to C_α we are allowed at each α, we obtain a hierarchy of principles of decreasing strength which can be used to test consistency strength of various principles (see more on this in [3]). In the light of these facts it is natural that the question of how to add or destroy a square principle by forcing has been a central theme. See [3] for a description of some of the many known results including versions of an older result of Magidor in which a square sequence is added by forcing.

The way that Magidor adds a square is to force by initial segments along a closed bounded subsets of the domain, and to use the existence of the "top" point in the domain of a forcing condition to show that the forcing is strategically closed. Note

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that the principle $\square_\omega$ is trivially true, by taking C_α to be any club of α of order type ω, so the first non-trivial instance of square is $\square_{\omega_1}$. Magidor's method means that to get $\square_{\omega_1}$ we need to force with conditions whose domain has size ω_1. In this work we have been interested to do this differently, using conditions whose domain is a finite set. The interest in doing this stems from a need to understand how one can control a one cardinal gap in forcing notions, which is a subject that has been of interest for various combinatorial issues for a long time. A glaring example of the need to develop this subject is the combinatorics of $(\omega_1^{\omega_1}, \leq_{\mathrm{Fin}})$, which in contrast with the vast body of knowledge about $(\omega^\omega, \leq_{\mathrm{Fin}})$ remains a mysterious object. An important development on the subject of $(\omega_1^{\omega_1}, \leq_{\mathrm{Fin}})$ is Koszmider's paper [9] in which he shows that it is consistent to have an increasing chain of length ω_2 in this structure. Koszmider's paper also gives an overview of the difficulties that there are in forcing one gap results.

Koszmider's method is to force with conditions where a morass is used as a side condition. Our method is more directly connected to a different approach, which was used to force a club on ω_2 using finite conditions. This was done in two different but similar ways by Friedman in [5] and Mitchell in [11]. Both approaches are built upon a version of adding a square on ω_1 using finite conditions, as discovered by Baumgartner [2] and modified by Abraham in [1]. The main idea in Baumgartner's approach is that to force a club in ω_1 and avoid problems at the limit stages, one needs to specify by each condition not only what will go in the club, but also whole intervals that need to stay out of it. At ω_2 one can do the same, but now one needs to add side conditions in the form of coherent systems of models in order to make sure that cardinals are preserved, as was first done by Todorčević in [15]. This already is technically rather involved. What we have done is add to this the coherent partial square sequence. Namely, we actually force a square indexed by a club sequence – the existence of such a square implies the existence of an actual square sequence. This club set is like the one added by Friedman and Mitchell. The actual forcing notion needs to take into account the coherence of the square sequence, and this is reflected in the complexity of the coherence conditions between the models which form part of the forcing conditions. An advantage of this type of approach over the morass-based approach is that it requires less from the ground model – for example Friedman's forcing only needs a weakening of CH in the ground model. We use the full CH together with $2^{\omega_1} = \omega_2$. The main difficulties of both approaches of course are the same, and they stem from the fact that combinatorics at ω_2 is much less prone to independence than the combinatorics at ω_1, as exemplified by the above mentioned result of Shelah on $\diamondsuit$ ([12]). It is both in developing combinatorics and fine forcing techniques that we can better understand the truth about ω_2.

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2. Preliminaries

Most of the notation is standard. Relation $A \subset B$ means that A is either a proper subset of B or equal to B. $|X|$ is the cardinality of set X. For a set of ordinals X, a limit point of X is an ordinal α such that $\alpha = \sup(Y)$ for some $Y \subset X$ or, equivalently, if $\alpha = \sup(X \cap \alpha)$. $\mathrm{Lim}(X)$ is a set of limit points of X. For a function f, $\mathcal{D}_f$ denotes the domain of f, and $f|_A$ denotes the restriction of f to the set $A \cap \mathcal{D}_f$. If α and β are ordinals then the interval (α, β) denotes the set

$\{\mu \mid \mu$ is an ordinal, $\alpha < \mu < \beta\} = \beta \setminus (\alpha + 1)$. Closed and half open interval are defined similarly. $[A]^\kappa$ is the set of all subsets of A of cardinality κ. Set $[A]^{\leq\kappa}$ is defined analogously.

For a regular cardinal θ, H_θ is the set of all sets x with hereditary cardinality less than θ (i.e. the transitive closure of x has cardinality less than θ). For $\theta > \omega_2$ we consider H_θ to be a model with the standard relation $\in$ and a fixed well-ordering $\leq^*$. We will primarily work with H_{ω_2} which we view as a model with $\in$ and $\leq^* \restriction H_{\omega_2}$. Cardinal θ is said to be *large enough* if every set in consideration is an element of H_θ.

Definition 2.1. *Suppose κ is a regular cardinal. A set $C \subset \kappa$ is called a* closed unbounded *set or a* club *in κ if:*

(1) for every increasing sequence $\langle \alpha_i \mid i < \lambda\rangle$ of elements from C, for some $\lambda < \kappa$, we have $\bigcup_{i<\lambda} \alpha_i \in C$ (closed)*;*

(2) for every $\alpha < \kappa$ there exists some $\beta \in C$ such that $\beta > \alpha$ (unbounded)*.*

The assumption that κ is a regular cardinal can be replaced by a singular cardinal or even an ordinal. In that case, λ from clause (1) has to be below $\mathrm{cf}(\kappa)$. In fact, clause (1) can be replaced by equivalent notion, that $\mathrm{Lim}(C) \cap \kappa \subset C$.

Definition 2.2. *Suppose κ is a regular cardinal.* Square principle $\square_\kappa$ *(square kappa) is a sequence $\langle C_\alpha \mid \alpha$ is a limit ordinal in $\kappa^+\rangle$ such that:*

(1) C_α is a club in α for every α;

(2) if $\alpha \in \mathrm{Lim}(C_\beta)$ then $C_\alpha = C_\beta \cap \alpha$ (coherence)*;*

(3) if $\mathrm{cf}(\alpha) < \kappa$ then $|C_\alpha| < \kappa$ (nontriviality)*.*

In case $\kappa = \omega_1$, the nontriviality clause simply stipulates that if $\mathrm{cf}(\alpha) = \omega$ then $|C_\alpha| = \omega$.

We shall call sequence $\langle C_\alpha \mid \alpha \in \mathcal{C}\rangle$ for some set $\mathcal{C} \subset \mathrm{Lim}(\kappa^+)$ a *square-like sequence* if it is fulfilling all three clauses of the definition of a square sequence.

3. Background on elementary submodels

A model M is an *elementary submodel* of a model N, $M \prec N$, if for every formula φ with parameters $a_1, \ldots, a_n \in M$, φ is true in M if and only if it is true in N. If M is a countable elementary submodel of H_θ for $\theta \geq \omega_1$ then $M \cap \omega_1$ is an ordinal denoted by δ_M. Also, if $|x| \leq \omega$ and $x \in M$ then $x \subset M$.

We begin by listing a few lemmas about elementary submodels which will come in handy later on. We add proofs for completeness. When dealing with elementary submodels, the Tarski-Vaught test [8] comes as a very useful tool.

Theorem 3.1 (Tarski-Vaught test)**.** *Let M be a submodel of N. Then M is elementary submodel of N if and only if for every formula $\phi(x, a_1, \ldots, a_n)$ and $a_1 \ldots, a_n \in M$, if $N \models \exists x \phi(x, a_1, \ldots, a_n)$ then there exists $b \in M$ such that $N \models \phi(b, a_1, \ldots, a_n)$.*

Lemma 3.2. *Suppose $N \prec H_\theta$ for some large enough θ. Then $N \cap H_{\omega_2} \prec H_{\omega_2}$.*

Proof. Let $a_1, \ldots, a_n \in N \cap H_{\omega_2}$ and suppose that $H_{\omega_2} \models \psi(a_1, \ldots, a_n)$ where ψ is the formula $\exists x \phi(x, a_1, \ldots, a_n)$. Then $\psi^{H_{\omega_2}}$—the relativization of ψ to H_{ω_2}—is true. Formula $\psi^{H_{\omega_2}}$ is equivalent to the formula ψ^* obtained by replacing every occurence of $\exists y \in H_{\omega_2}\ \chi(y, \ldots)$ with $\exists y(\chi(y, \ldots) \wedge |\operatorname{tr\,cl}(y)| \leq \omega_1)$, and similarly for the

universal quantifier. We get ϕ^* from ϕ in the same way. Now, $H_\theta \models \psi^*(a_1, \dots, a_n)$, or in other words, $H_\theta \models \exists x(\phi^*(x, a_1, \dots, a_n) \wedge |\operatorname{tr}\operatorname{cl}(x)| \leq \omega_1)$.

Since $\omega_1 \in N$, by Tarski-Vaught test there exists some $b \in N$ such that $H_\theta \models \phi^*(b, a_1, \dots, a_n) \wedge |\operatorname{tr}\operatorname{cl}(b)| \leq \omega_1$. Hence, there exists $b \in N \cap H_{\omega_2}$ such that $H_\theta \models \phi^{H_{\omega_2}}(b, a_1, \dots, a_n)$, and as a consequence, $H_{\omega_2} \models \phi(b, a_1 \dots, a_n)$, which by Tarski-Vaught test means that $N \cap H_{\omega_2} \prec H_{\omega_2}$. $\surd$

Lemma 3.3. *Suppose $N, M \prec (H_{\omega_2}, \in, \leq^*)$. Then $N \cap M \prec (H_{\omega_2}, \in, \leq^*)$.*

Proof. Let $a_1, \dots, a_n \in N \cap M$ and suppose that $H_{\omega_2} \models \exists x \phi(x, a_1, \dots, a_n)$. Let $\psi(x, a_1, \dots, , a_n)$ be the formula $\phi(x, a_1, \dots, a_n) \wedge \forall y(\phi(y, a_1, \dots, a_n) \to x \leq^* y)$. Then $H_{\omega_2} \models \exists x \psi(x, a_1, \dots, a_n)$. By Tarski-Vaught test there exist $x_1 \in M$ and $x_2 \in N$ such that $H_{\omega_2} \models \psi(x_1, a_1, \dots, a_n)$ and $H_{\omega_2} \models \psi(x_2, a_1, \dots, a_n)$. But then $x_1 = x_2 =: x^* \in M \cap N$, and $H_{\omega_2} \models \phi(x^*, a_1, \dots, a_n)$. By Tarski-Vaught test, $M \cap N \prec H_{\omega_2}$. $\surd$

Lemma 3.4. *If $M \prec H_\kappa$ for some $\kappa > \omega_1$, and $\sup(M \cap \alpha) < \alpha$ for some ordinal $\alpha \in M$, then $\operatorname{cf}(\alpha) > \omega$.*

Proof. If $\operatorname{cf}(\alpha) = \omega$ then there is a cofinal function $f : \omega \to \alpha$ in M, hence $\sup(M \cap \alpha) = \alpha$, a contradiction. $\surd$

Lemma 3.5. *Let $M, N \prec H_\kappa$ for some $\kappa > \omega_1$, and suppose that $M \in N$. If $\alpha \notin N$ then $\sup(M \cap \alpha) < \sup(N \cap \alpha)$.*

Proof. If $\alpha \geq \sup(N)$ then $\sup(M \cap \alpha) = \sup(M) < \sup(N) = \sup(N \cap \alpha)$. Suppose now that $\alpha < \sup(N)$ and let $\beta := \sup(M \cap \alpha)$ and $\beta' := \min(N \setminus \alpha) \in N$. Since $M \subset N$, $\beta = \sup(M \cap \beta')$. Hence, by elementarity, $\beta \in N$, and therefore $\beta < \sup(N \cap \alpha)$. $\surd$

The standard reference for basic set-theoretic notions and facts is [6]. Additional source for results on elementary models in a very concise form is [4], as well as [8].

In our application of elementary submodels we will basically only be interested in the ordinals that lie inside them. To simplify the notation we will write $\mathscr{M}$ for a model and M for its set of ordinals $\mathscr{M} \cap Ord$. We will use the term "model" for both $\mathscr{M}$ and M.

4. Forcing a square

Let V be some countable transitive model of (a sufficiently large finite fragment of) ZFC together with CH and "$2^{\omega_1} = \omega_2$". Since we want to force the existence of a square sequence, the working part of forcing notion P will consist of finite partial square sequences. We will add *safeguards* which will help us separate clubs from condition q and clubs from restriction $p \leq q$. This will be instrumental in the proof of properness.

It should be noted once again that we do not have to build a square sequence on the whole $\operatorname{Lim}(\omega_2)$. Instead, it is enough for the domain of the built sequence to be a club in ω_2, because we can always extend a square sequence from a club to the full $\operatorname{Lim}(\omega_2)$ (see Lemma 5.13). This is the reason why we add intervals as a part of conditions. These intervals will serve as gaps in what will ultimately be the desired club in $\operatorname{Lim}(\omega_2)$. This way of forcing a club was introduced by Baumgartner in [2].

Before we are ready to present the definition of forcing we have to define a few auxiliary notions. For $\alpha < \omega_2$, $\operatorname{cf}(\alpha) = \omega_1$, let $\mathscr{E}_\alpha$ denote some fixed countable set of

clubs in α of order type ω_1, and $\mathscr{E} := \langle \mathscr{E}_\alpha \mid \alpha < \omega_2 \rangle$. Define $\mathfrak{M}_0 := \{\mathscr{M} \prec H_{\omega_2} \mid \mathscr{M}$ is countable, $\mathscr{E}_\alpha \in \mathscr{M}$ for every $\alpha \in \mathscr{M}$ with $\mathrm{cf}(\alpha) = \omega_1\}$. The set $\mathfrak{M}_0$ will act as a pool of possible side conditions.

For a large enough cardinal θ let $\mathfrak{M}_1 := \{\mathscr{M} \prec H_\theta \mid \mathscr{M}$ is countable, $\mathscr{E} \in \mathscr{M}\}$. Then $\mathfrak{M}_1$ is a stationary set in $[H_\theta]^\omega$. Also, if $\mathscr{N} \in \mathfrak{M}_1$ and $\alpha \in \mathscr{N}$ then, by elementarity, $\mathscr{E}_\alpha \in \mathscr{N}$. If $\alpha < \omega_2$ then $\mathscr{E}_\alpha \in H_{\omega_2}$, hence $\mathscr{N} \cap H_{\omega_2} \in \mathfrak{M}_0$.

Definition 4.1. *If $\mathscr{M}_1, \mathscr{M}_2 \prec H_{\omega_2}$ are countable, then we say that the sets M_1 and M_2 are* compatible *if the following two clauses hold as stated and with M_1 and M_2 switched:*

(a) either $M_1 \cap M_2 \in \mathscr{M}_1$ if $\sup(M_1 \cap M_2) \in \mathscr{M}_1$, or $[\delta]^\omega \cap \mathscr{M}_1 \subset \mathscr{M}_2$ where $\delta := \sup(M_1 \cap M_2)$ if $\sup(M_1 \cap M_2) \notin \mathscr{M}_1$;

(b) $\{\min(M_1 \setminus \lambda) \mid \lambda \in M_2, \sup(M_1 \cap M_2) < \lambda < \sup(M_1)\} \cup \{\min(M_1 \setminus \sup(M_1 \cap M_2))\}$ is a finite subset of M_1.

The set in (b) is called the set of M_1-*fences for* M_2. This definition of compatibility between elementary submodels (or in this case their sets of ordinals) is due to Mitchell [11]. In fact, this version is a slight strengthening of Mitchell's compatibility condition. The need for a slightly stronger version stems from the fact that we have to work with sets of ordinals, namely clubs, instead of just ordinals. Actually, workng with sets of ordinals adds a whole new level of difficulty to the forcing construction and most of the effort had to be invested to this end.

The following simple lemma shows that our version of compatibility between two models is indeed stronger than Mitchell's version.

Lemma 4.2. *If $[\delta]^\omega \cap \mathscr{M}_1 \subset \mathscr{M}_2$ then $M_1 \cap M_2$ is an initial segment of M_1, i.e. $M_1 \cap M_2 = M_1 \cap \delta$.*

Proof. Consider $\alpha \in M_1 \cap \delta$, and let A be some ω-sequence of ordinals smaller than α. Then $\{\alpha\} \cup A \in [\delta]^\omega \cap \mathscr{M}_1$, hence $\{\alpha\} \cup A \in \mathscr{M}_2$. Therefore $\alpha = \max(\{\alpha\} \cup A) \in M_2$. $\surd$

In lieu of the above lemma, we will say that the intersection $M_1 \cap M_2$ is an initial segment of M_1 whenever clause (a) of Definition 4.1 holds.

Remark 4.3. If M_1 and M_2 are compatible then their structure vis-a-vis each other is particularly simple. Above $\sup(M_1 \cap M_2)$ they consist of finitely many (not necessarily continuous) interchanging blocks, as witnessed by both fences. Below, they are either equal or one is a subset of the other. Namely, if there exist $\lambda_1 \in M_1 \setminus M_2$ and $\lambda_2 \in M_2 \setminus M_1$, $\lambda_1, \lambda_2 < \sup(M_1 \cap M_2)$, then $M_1 \cap M_2$ is not an initial segment of either of them, hence it is an element of both $\mathscr{M}_1$ and $\mathscr{M}_2$. Therefore, $M_1 \cap M_2 \in \mathscr{M}_1 \cap \mathscr{M}_2$, and hence, $\sup(M_1 \cap M_2) \in \mathscr{M}_1 \cap \mathscr{M}_2$. But then $\sup(M_1 \cap M_2) + 1 \in \mathscr{M}_1 \cap \mathscr{M}_2$, which is obviously a contradiction.

Definition 4.4. *The forcing notion P is a set of conditions of the form $p := (\mathcal{F}_p,\ \mathcal{S}_p,\ \mathcal{O}_p,\ \mathcal{M}_p)$, where*

(1) $\mathcal{F}_p : \mathrm{Lim}(\omega_2) \to \mathcal{P}(\omega_2)$, $|\mathcal{F}_p| < \omega$, $\mathcal{F}_p(\alpha)$ is a club $C_\alpha \subset \alpha$ of order type $\leq \omega_1$ for all $\alpha \in \mathcal{D}_p := \mathrm{dom}(\mathcal{F}_p)$, and if $\mathrm{cf}(\alpha) = \omega_1$ then $C_\alpha \in \{C \setminus \beta \mid C \in \mathscr{E}_\alpha, \beta \in \mathcal{D}_p \cap \alpha\}$;

(2) $\mathcal{S}_p \subset \mathcal{D}_p$ and $\alpha \in \mathcal{S}_p$ for every $\alpha \in \mathcal{D}_p$ with $\mathrm{cf}(\alpha) = \omega_1$;

(3) $\mathcal{M}_p$ is a finite set such that if $M \in \mathcal{M}_p$ then there exists a countable elementary submodel $\mathscr{M} \in \mathfrak{M}_0$ with $M = \mathscr{M} \cap Ord$, and additionally, $\sup(M) \in \mathcal{S}_p$ for every $M \in \mathcal{M}_p$;

(4) for every $\alpha, \beta \in \mathcal{D}_p$, $\alpha \neq \beta$, if $\mu \in \mathrm{Lim}(C_\alpha) \cap \mathrm{Lim}(C_\beta)$ then $C_\alpha \cap \mu = C_\beta \cap \mu$;

(5) if $\alpha \in \mathcal{D}_p$ and $\sigma \in \mathcal{S}_p$, $\sigma < \alpha$, then $C_\alpha \cap \sigma$ is a finite set;

(6) for all $\alpha \in \mathcal{D}_p$ and $M \in \mathcal{M}_p$:

(a) if $\alpha \in M$ then $C_\alpha \in \mathscr{M}$,

(b) if $\alpha \notin M$ is such that $\alpha < \sup(M)$, or if $\alpha \in M$ is such that $\sup(M \cap \alpha) < \alpha$, then $\min(M \setminus \alpha) \in \mathcal{S}_p$ and $\sup(M \cap \alpha) \in \mathcal{D}_p$,

(c) if $\alpha \notin M$ and $\sup(M \cap \alpha) < \alpha < \sup(M)$ then $C_\alpha = C_\beta \cap \alpha$ if there is some $\beta \in \mathcal{D}_p$, $\beta > \alpha$, such that $\alpha \in \mathrm{Lim}(C_\beta)$, otherwise $C_\alpha \cap \sup(M \cap \alpha)$ is a finite set,

(d) if $\alpha \notin M$ and $\sup(M \cap \alpha) = \alpha$ then $C_\alpha = C_\beta \cap \alpha$ if there is some $\beta \in \mathcal{D}_p$, $\beta > \alpha$, such that $\alpha \in \mathrm{Lim}(C_\beta)$, otherwise C_α is some cofinal sequence in α of length ω;

(7) $\mathcal{O}_p$ is a finite set of half open nonempty intervals $(\beta', \beta] \subset \omega_2$ such that $\mathcal{D}_p \cap \bigcup \mathcal{O}_p = \emptyset$;

(8) if $(\beta', \beta] \in \mathcal{O}_p$ and $M \in \mathcal{M}_p$ then either $(\beta', \beta] \in \mathscr{M}$ or $(\beta', \beta] \cap \mathscr{M} = \emptyset$;

(9) if $M_1, M_2 \in \mathcal{M}_p$ then they are compatible, and the M_1-fence for M_2 and the M_2-fence for M_1 are both subsets of $\mathcal{S}_p$.

For $p, q \in P$ define $p \leq q \overset{def}{\iff} \mathcal{F}_p \subset \mathcal{F}_q,\ \mathcal{S}_p \subset \mathcal{S}_q,\ \mathcal{O}_p \subset \mathcal{O}_q,\ \mathcal{M}_p \subset \mathcal{M}_q$.

Clause (6b) tells us that a gap in a model M has to be closed from above by a safeguard if there is something (i.e. an ordinal $\alpha \in \mathcal{D}_p$) inside that gap. This safeguard is an echo of α resonating in M, warning everybody in M to stay away from that gap. Fences from clause (9) serve exactly the same purpose.

Notice that in clause (8), the interval $(\beta', \beta]$ is an element of the model $\mathscr{M}$ if and only if both β' and β are in M.

Lemma 4.5. $(P, \leq)$ *is a separative forcing notion.*

Proof. Transitivity is trivial. The minimal element is $(\emptyset, \emptyset, \emptyset, \emptyset)$. For separativeness consider an arbitrary condition $p \in P$. We will find two incompatible extensions. Let $\alpha := \sup(\mathcal{D}_p \cup \bigcup \mathcal{O}_p \cup \bigcup \mathcal{M}_p)$, and $\beta := \alpha + \omega < \omega_2$. Define $C_\beta := [\alpha, \beta)$ and $C'_\beta := (\alpha, \beta)$. It is easy to check that $q := (\mathcal{F}_p \cup \{(\beta, C_\beta)\}, \mathcal{S}_p, \mathcal{O}_p, \mathcal{M}_p)$ and $q' := (\mathcal{F}_p \cup \{(\beta, C'_\beta)\}, \mathcal{S}_p, \mathcal{O}_p, \mathcal{M}_p)$ are both conditions extending p, and that they are incompatible. Notice, that since $\mathrm{cf}(\beta) = \omega$, C_β and C'_β need not be in $\mathscr{E}_\beta$. $\surd$

We first prove several lemmas that show us a little bit more about the structure of conditions in P, and will be helpful in further proofs. Most notably, they will shed some lights on the correspondence between models and clubs, and thus clarify clause (6).

Lemma 4.6. *Let $p \in P$, and suppose that $\alpha, \gamma \in \mathcal{D}_p$ and $M \in \mathcal{M}_p$ are such that $\alpha < \sup(M)$, $\alpha \notin M$, and $\alpha \in \mathrm{Lim}(C_\gamma)$. Then $\gamma \leq \min(M \setminus \alpha)$.*

Proof. By (6b), $\sigma := \min(M \setminus \alpha) \in \mathcal{S}_p$, hence if $\gamma > \sigma$ then, by (5), C_γ has no limit points below σ. $\surd$

Notice that if $\alpha \in \mathrm{Lim}(C_\gamma)$ then $\mathrm{cf}(\alpha) = \omega$, otherwise C_γ would have order type larger than ω_1.

Lemma 4.7. *If p, α, γ and M are as in the previous lemma and $\max\{\gamma \in \mathcal{D}_p \mid \alpha \in \mathrm{Lim}(C_\gamma)\} < \min(M \setminus \alpha)$, or if $\alpha > \sup(M)$, then $C_\alpha \cap \sup(M \cap \alpha)$ is finite, and therefore $\sup(M \cap \alpha) < \alpha$.*

Proof. If $\alpha > \sup(M)$ then the conclusion follows from clauses (3) and (5). Suppose now that $\alpha < \sup(M)$ and $\alpha \in \mathrm{Lim}(C_\gamma)$. Then $\gamma \notin M$ and $\sup(M \cap \gamma) > \gamma$. If γ is not a limit point of any $C_{\gamma'}$ then, by (6c), $C_\gamma \cap \sup(M \cap \gamma)$ is finite. Since $C_\alpha \subset C_\gamma$ and $\sup(M \cap \alpha) = \sup(M \cap \gamma)$, $C_\alpha \cap \sup(M \cap \alpha)$ is also finite. If $\gamma \in \mathrm{Lim}(C_{\gamma'})$ then $\alpha \in \mathrm{Lim}(C_{\gamma'})$ and we can repeat the above argument. As $|\mathcal{D}_p| < \omega$, we only have to repeat it finitely many times, and in the end we can conclude that $C_\alpha \cap \sup(M \cap \alpha)$ is finite. $\surd$

Lemma 4.8. *If $M \in \mathcal{M}_p$ is countable then $C_{\sup(M)}$ is an ω-sequence.*

Proof. By clauses (3) and (5), $\sup(M)$ cannot be a limit point of any C_γ for $\gamma \in \mathcal{D}_p$. So $C_{\sup(M)}$ is an ω-sequence by clause (6d). $\surd$

Lemma 4.9. *Let $\mathscr{N}' \in \mathfrak{M}_1$ be a countable elementary submodel of H_θ. If p is a condition in $P \cap \mathscr{N}'$ then there exists an extension $q \geq p$ such that $\mathscr{N}' \cap H_{\omega_2} \in \mathcal{M}_q$.*

Proof. Let p be of the form $(\mathcal{F}_p, \mathcal{S}_p, \mathcal{O}_p, \mathcal{M}_p)$ and let $\mathscr{N} := \mathscr{N}' \cap H_{\omega_2} \in \mathfrak{M}_0$. By Lemma 3.2, $\mathscr{N} \prec H_{\omega_2}$. Note also that $p \in \mathscr{N}$.

For every $\alpha \notin N$ such that $\alpha = \sup(N \cap \gamma)$ for some $\gamma \in \mathcal{D}_p$, let C_α be a club according to clause (6d). In the case of $\alpha \in \mathrm{Lim}(C_\beta)$ for some $\beta \in \mathcal{D}_p$ this choice is well-defined because by clause (4) it does not depend on β. It is worth mentioning that $\mathrm{cf}(\alpha) = \omega$, hence C_α need not be in $\mathscr{E}_\alpha$. Notice, that by Lemma 3.4, $\mathrm{cf}(\gamma) = \omega_1$. Therefore, γ is already in $\mathcal{S}_p$ and does not have to be added for clause (6b) to be satisfied. We will also have to add $\sup(N)$ to the set of safeguards. For the corresponding club $C_{\sup(N)}$ we pick any cofinal ω-sequence. Again, $\mathrm{cf}(\sup(M)) = \omega$, therefore $C_{\sup(M)}$ des not have to be in $\mathscr{E}_{\sup(M)}$.

Define $q := (\mathcal{F}_p \cup \{(\alpha, C_\alpha) \mid \alpha \notin N, \alpha = \sup(N \cap \gamma) \text{ for some } \gamma \in \mathcal{D}_p\} \cup \{(\sup(N), C_{\sup(N)})\}, \mathcal{S}_p \cup \{\sup(N)\}, \mathcal{O}_p, \mathcal{M}_p \cup \{N\})$. Most of the clauses of definition 4.4 are trivially true, including clause (6), which is due to the fact that we used clause (6) to construct additional clubs. These new clubs conform to clauses (1) and (4) as well, since clause (6d) was added specifically for this purpose. Notice, that for $M \in \mathcal{M}_p$ the M-fence for N is the empty set, while the N-fence for M is $\{\sup(M \cap N)\} = \{\sup(M)\}$ which is a subset of $\mathcal{S}_p \subset \mathcal{S}_q$ by clause (3). As for clause (7), suppose that some newly added $\alpha < \sup(N)$ falls into some interval $(\beta', \beta]$. Then its corresponding $\gamma \in \mathcal{D}_p$ was already in this interval, since $\{\beta', \beta\} \subset N$. But that is in a contradiction with clause (7) in p.

Hence q is the desired condition extending p. $\surd$

Lemma 4.10. *Let $\mathscr{N} \in \mathfrak{M}_1$ be a countable elementary submodel of H_{ω_2}, and suppose that $r \in P \setminus \mathscr{N}$ is such that $N \in \mathcal{M}_r$. Then $r_{\mathscr{N}} := (\mathcal{F}_r \cap \mathscr{N}, \mathcal{S}_r \cap \mathscr{N}, \mathcal{O}_r \cap \mathscr{N}, (\mathcal{M}_r \cap \mathscr{N}) \cup \{M \cap N \mid M \in \mathcal{M}_r, M \notin \mathscr{N}, M \cap N \in \mathscr{N}\})$ is a condition in $P \cap \mathscr{N}$.*

Proof. First note that by clause (6a), $\mathcal{D}_{r_{\mathscr{N}}} = \mathcal{D}_r \cap \mathscr{N}$. Also, by Lemma 3.3, $\mathscr{M} \cap \mathscr{N} \prec H_{\omega_2}$. Additionally, $\mathscr{M} \cap \mathscr{N} \in \mathfrak{M}_0$, hence $M \cap N$ can be added to $\mathcal{M}_{r_{\mathscr{N}}}$. Notice that if $M \cap N \in \mathscr{N}$ then $\sup(M \cap N) \in \mathcal{S}_r \cap \mathscr{N}$ because it is in the N-fence for M, hence clause (3) is satisfied.

It is obvious that $r_{\mathscr{N}} \in \mathscr{N}$. Slightly less trivial thing to prove is that $r_{\mathscr{N}} \in P$. Compatibility between the elements of $\mathcal{F}_r$, $\mathcal{S}_r$, $\mathcal{O}_r$ and $\mathcal{M}_r$ is inherited from r, as is the compatibility between two models from $\mathcal{M}_r$. The same can be said for the compatibility between $\mathcal{F}_{r_{\mathscr{N}}}$, $\mathcal{O}_{r_{\mathscr{N}}}$ and a model of the form $M \cap N \in \mathcal{M}_{r_{\mathscr{N}}}$, however with a closer inspection of clause (6).

For clause (6b) consider $\alpha \in \mathcal{D}_{r_{\mathscr{N}}}$ and $M \cap N \in \mathcal{M}_{r_{\mathscr{N}}}$ such that $\alpha \notin M \cap N$. That means that $\alpha \notin M$. Since $M \cap N \in \mathscr{N}$, $M \cap N$ is an initial segment of M. If $\alpha < \sup(M \cap N)$ then $\min((M \cap N) \setminus \alpha) = \min(M \setminus \alpha) \in \mathcal{S}_r \cap \mathscr{N}$ by clause (6b) in r, hence $\min((M \cap N) \setminus \alpha) \in \mathcal{S}_{r_{\mathscr{N}}}$. By the same argument, $\sup((M \cap N) \cap \alpha) \in \mathcal{D}_{r_{\mathscr{N}}}$. Similarly, if $\alpha \in M \cap N$ such that $\sup((M \cap N) \cap \alpha) < \alpha$ then by (6b) in r, $\alpha = \min((M \cap N) \setminus \alpha) \in \mathcal{S}_{r_{\mathscr{N}}}$ and $\sup((M \cap N) \cap \alpha) \in \mathcal{D}_{r_{\mathscr{N}}}$.

For clause (6c) assume that $\alpha \notin M \cap N$ is such that $\sup((M \cap N) \cap \alpha) < \alpha < \sup(M \cap N)$. The only potential problem is if $\alpha \in \mathrm{Lim}(C_\beta)$ for some $\beta \in \mathcal{D}_r$ while $\alpha \notin \bigcup_{\gamma \in \mathcal{D}_{r_{\mathscr{N}}}} \mathrm{Lim}(C_\gamma)$. In such a case, by Lemma 4.6, $\beta \leq \min(M \setminus \alpha)$. In fact, $\beta < \min(M \setminus \alpha)$ because $\min(M \setminus \alpha) \in N$. Then by Lemma 4.7, $C_\alpha \cap \sup(M \cap \alpha)$ is finite.

For clause (6d) assume that α is a supremum of a block of $M \cap N$. Let us ask ourselves a question. Is it possible that α is a limit point of some C_β in r but not in $r_{\mathscr{N}}$? It is only possible if $\alpha = \sup(M \cap N)$. Otherwise, as above, $\min(M \setminus \alpha) \in \mathcal{S}_r$ which makes it the only candidate for β. However, this β and C_β remain in $r_{\mathscr{N}}$ and do not answer our question affirmatively. On the other hand $\sup(M \cap N) \in \mathcal{S}_r$ by (9) in r and hence by (5) cannot be a limit point of any C_β, neither in r nor in $r_{\mathscr{N}}$.

The thing that merits the closest attention is the compatibility between two models of the form $M \cap N \in \mathcal{M}_{r_{\mathscr{N}}}$. Consider $M'_i := M_i \cap N$, $i = 1, 2$, such that $M_i \in \mathcal{M}_r \setminus \mathscr{N}$ and $M'_i \in \mathscr{N}$. Let x_1 be the M_1-fence for M_2. Then $x_1 \cap N = x_1 \cap \sup(M_1 \cap N)$ is the M'_1-fence for M'_2. Here we use the fact that $M_1 \cap N$ is an initial segment of M_1. We get the M'_2-fence for M'_1 in a similar way.

Suppose that $M_1 \cap M_2 \in \mathscr{M}_2$. Let $\lambda := \sup(M'_1 \cap M'_2) \in N$. We have to consider three cases.

Case 1: $\lambda \in M_2$. Then, by elementarity, $M'_1 \cap M'_2 = M_1 \cap M_2 \cap \lambda \in \mathscr{M}_2 \cap \mathscr{N} = \mathscr{M}'_2$.

Case 2: $\lambda \notin M_2$ and $\lambda \in M_1$. Obviously $\lambda \leq \sup(M_1 \cap M_2)$. If $\lambda < \sup(M_1 \cap M_2)$ then $\lambda \in M_2$, because $M_1 \cap M_2$ is an initial segment of M_1. Hence we get a contradiction. If $\lambda = \sup(M_1 \cap M_2)$ then $\lambda \in M_2$, because $M_1 \cap M_2 \in \mathscr{M}_2$, and again we get a contradiction.

Case 3: $\lambda \notin M_2$ and $\lambda \notin M_1$. If $\lambda = \sup(M_1 \cap M_2)$ then, as above, $\lambda \in M_2$. If $\lambda < \sup(M_1 \cap M_2)$ then $\lambda = \sup((M_1 \cap M_2) \cap \min(M_2 \setminus \lambda)) \in M_2$ by elementarity. Both possibilities lead to a contradiction.

Suppose now that $[\mu]^\omega \cap \mathscr{M}_2 \subset \mathscr{M}_1$ where $\mu := \sup(M_1 \cap M_2)$. If $A \in [\lambda]^\omega$ is in $\mathscr{M}'_2$ then $A \in \mathscr{M}_2 \cap \mathscr{N}$ and $A \in [\mu]^\omega$, hence $A \in \mathscr{M}_1 \cap \mathscr{N} = \mathscr{M}'_1$. So $[\lambda]^\omega \cap \mathscr{M}'_2 \subset \mathscr{M}'_1$.

Basically, $M'_1 \cap M'_2$ is in the same correspondence with both $\mathscr{M}'_1$ and $\mathscr{M}'_2$ as $M_1 \cap M_2$ is with $\mathscr{M}_1$ and $\mathscr{M}_2$ respectively. $\surd$

We are now ready to prove the most important facet of forcing P, namely the fact that it preserves ω_1. We do that by proving that P is proper. There are several equivalent definitions of properness. We shall use the following one.

Definition 4.11. *Let P be a forcing notion, θ a large enough cardinal, and let $\mathfrak{N}$ be some set of countable elementary submodels of H_θ, stationary in $[H_\theta]^\omega$.*

(1) Condition $q \in P$ is $\mathscr{N}$-generic *if for every extension $r \geq q$, $r \in P$, and every dense set $\mathscr{D} \subset P$, $\mathscr{D} \in \mathscr{N}$, there exists some condition $s \in \mathscr{D} \cap \mathscr{N}$ which is compatible with r.*

(2) P is proper *if for every $\mathscr{N} \in \mathfrak{N}$ such that $P \in \mathscr{N}$, every condition $p \in P \cap \mathscr{N}$ has an $\mathscr{N}$-generic extension.*

Proposition 4.12. *Forcing P is proper.*

Proof. Let θ be a large enough cardinal. Fix a countable elementary submodel $\mathscr{N}' \prec H_\theta$, $\mathscr{N}' \in \mathfrak{M}_1$, such that $P \in \mathscr{N}'$, and consider an arbitrary $p = (\mathcal{F}_p, \mathcal{S}_p, \mathcal{O}_p, \mathcal{M}_p) \in P \cap \mathscr{N}'$. Define $\mathscr{N} := \mathscr{N}' \cap H_{\omega_2} \in \mathfrak{M}_0$ and let q be an extension of p given by Lemma 4.9. We will prove that q is an $\mathscr{N}'$-generic extension of p.

Suppose $r \in P$ is an arbitrary extension of q. Let $r_{\mathscr{N}}$ be the condition given by Lemma 4.10. Proceed by fixing a dense open subset $\mathscr{D} \subset P$, $\mathscr{D} \in \mathscr{N}'$, and extend $r_{\mathscr{N}}$ to $s \in \mathscr{D} \cap \mathscr{N}'$. Clearly $s \in H_{\omega_2}$, hence $s \in \mathscr{N}$. We shall prove that r and s are compatible by proving clause by clause of Definition 4.4 that (a certain extension of) $t := (\mathcal{F}_r \cup \mathcal{F}_s, \mathcal{S}_r \cup \mathcal{S}_s, \mathcal{O}_r \cup \mathcal{O}_s, \mathcal{M}_r \cup \mathcal{M}_s)$ is a condition.

Clauses (1), (2) and (3) are obviously true.

Clause (4): take arbitrary $\alpha, \beta \in \mathcal{D}_t$. We can assume WLOG that $\alpha \in \mathcal{D}_r \setminus \mathcal{D}_s$ and $\beta \in \mathcal{D}_s \setminus \mathcal{D}_r$. If $\beta > \alpha$ then there are two possibilities. If $\beta = \min(N \setminus \alpha)$ then $\beta \in \mathcal{S}_r \subset \mathcal{D}_r$ by (6b) in r, which we assumed was not the case. If $\beta > \min(N \setminus \alpha)$ then $C_\beta \cap C_\alpha \subset C_\beta \cap \min(N \setminus \alpha)$ which is finite by (5) in s, because $\min(N \setminus \alpha) \in \mathcal{S}_r \cap \mathscr{N} \subset \mathcal{S}_s$ by (6b) in r. Hence, $\mathrm{Lim}(C_\beta) \cap \mathrm{Lim}(C_\alpha) = \emptyset$.

If $\beta < \sup(N) \leq \alpha$ then $C_\alpha \cap \beta$ is finite either by Lemma 4.8 or clause (5) in r, hence $\mathrm{Lim}(C_\alpha) \cap \mathrm{Lim}(C_\beta) = \emptyset$. If $\beta < \alpha < \sup(N)$, then by clause (6c) or (6d) in r, either $C_\alpha \cap \delta$ is finite for every $\delta < \sup(N \cap \alpha)$ including β, or $C_\alpha = C_{\gamma'} \cap \alpha$ for some $\gamma' \in \mathcal{D}_r$. In the latter case let γ be the largest such γ'. By Lemma 4.6, $\gamma \leq \min(N \setminus \alpha)$. If $\gamma < \min(N \setminus \alpha)$ then $C_\alpha \cap \beta$ is finite by Lemma 4.7, because $\beta < \sup(N \cap \alpha)$. If $\gamma = \min(N \setminus \alpha) \in \mathcal{D}_s$ and if C_α and C_β have a common limit point μ, then $\mu \in \mathrm{Lim}(C_\gamma)$, hence $C_\beta \cap \mu = C_\gamma \cap \mu = C_\alpha \cap \mu$. The first equality follows from (4) in s and the second follows from (4) in r.

If $\sup(N \cap \alpha) = \alpha$ then the argument is similar. If C_α is an ω-sequence of ordinals then (4) is trivially true. If $C_\alpha = C_\gamma \cap \alpha$ for some $\gamma \in \mathcal{D}_r$ and if $\alpha < \sup(N)$, then $\gamma = \min(N \setminus \alpha)$. Notice that if γ were below $\min(N \setminus \alpha)$ then C_α would be finite by (6c) in r. Now we get (4) in t just as above. If $\alpha = \sup(N)$ then C_α were again finite by (6c) or even (5) in r.

Clause (5): first consider arbitrary $\alpha \in \mathcal{D}_r \setminus \mathcal{D}_s$ and $\sigma \in \mathcal{S}_s \setminus \mathcal{S}_r$, $\sigma < \alpha$. By (6c) or (6d) in r, either $C_\alpha = C_\gamma \cap \alpha$ for some $\gamma \in \mathcal{D}_r$, or $C_\alpha \cap \delta$ is a finite set for every $\delta < \sup(N \cap \alpha)$. The second case is trivial, because $\sigma \in N$, hence $\sigma < \sup(N \cap \alpha)$. In the first case, if $\alpha < \sup(N)$ then we can assume that $\gamma = \min(N \setminus \alpha) \in \mathcal{D}_s$. Hence, $C_\alpha \cap \sigma = C_\gamma \cap \sigma$ which is a finite set by (5) in s. If $\alpha > \sup(N)$ then $C_\alpha \cap \sup(N)$ is finite regardless of γ, and if $\alpha = \sup(N)$ then C_α is an ω-sequence.

If $\alpha \in \mathcal{D}_s \setminus \mathcal{D}_r$ and $\sigma \in \mathcal{S}_r \setminus \mathcal{S}_s$, $\sigma < \alpha$, then $\min(N \setminus \sigma) \in \mathcal{S}_r \cap \mathscr{N} \subset \mathcal{S}_s$. Also, $\alpha \geq \min(N \setminus \sigma)$, but $\alpha \neq \min(N \setminus \sigma)$, otherwise $\alpha \in \mathcal{S}_r \subset \mathcal{D}_r$. Hence $\alpha > \min(N \setminus \sigma) \in \mathcal{S}_s$ and therefore $C_\alpha \cap \sigma \subset C_\alpha \cap \min(N \setminus \sigma)$ which is a finite set by (5) in s.

Clause (6): first consider arbitrary $\alpha \in \mathcal{D}_r \setminus \mathcal{D}_s$ and a $M \in \mathcal{M}_s \setminus \mathcal{M}_r$. Then $\alpha \notin N \supset M$ and $\sup(M \cap \alpha) < \alpha$ by Lemma 3.5. We argue just like for clause (5). Either $C_\alpha = C_\gamma \cap \alpha$ for some $\gamma \in \mathcal{D}_r \subset \mathcal{D}_t$, and we get (6c) in t, or $C_\alpha \cap \delta$ is a finite set for every $\delta < \sup(N \cap \alpha)$. Since $\sup(M \cap \alpha) < \sup(N \cap \alpha)$, we also get (6c).

Additionally, if $\alpha < \sup(M)$ then let $\sigma := \min(N \setminus \alpha) \in \mathcal{D}_s$. Since $M \subset N$, we have $\sup(M \cap \sigma) < \sigma$, hence, by (6b) in s, $\sup(M \cap \alpha) = \sup(M \cap \sigma) \in \mathcal{D}_s \subset \mathcal{D}_t$, and $\min(M \setminus \alpha) = \min(M \setminus \sigma) \in \mathcal{S}_s \subset \mathcal{S}_t$.

Now consider arbitrary $\alpha \in \mathcal{D}_s \setminus \mathcal{D}_r$ and $M \in \mathcal{M}_r \setminus \mathcal{M}_s$. Assume first that $\alpha \notin M$. If $\sup(M \cap N) < \alpha < \sup(M)$ then $\alpha' := \min(M \setminus \alpha) \in \mathcal{S}_r \subset \mathcal{S}_t$ by (9) in r

because there is a block of N inside a gap of M. Also, by (6b) in r, $\sup(M\cap\alpha) = \sup(M\cap\alpha') \in \mathcal{D}_r \subset \mathcal{D}_t$. Additionally (and also if $\alpha > \sup(M)$), if there is no $\delta \in \mathcal{D}_t$ such that $\alpha \in \mathrm{Lim}(C_\delta)$ then $C_\alpha \cap \sup(M\cap\alpha)$ is finite, because either there is some $\gamma \in M$, $\sup(M\cap N) < \gamma < \alpha$, in which case $\sigma := \min(N \setminus \gamma) \in \mathcal{S}_r \cap N \subset \mathcal{S}_s$ by (9) in r, or else $\sigma := \min(N\setminus \sup(M\cap N)) \in \mathcal{S}_r \cap N$, because σ is in the N-fence for M. In both cases $C_\alpha \cap \sup(M\cap\alpha) \subset C_\alpha \cap \sigma$ which is finite by (5) in s. Hence, we got (6c) which we had to get, because $\sup(M\cap\alpha) < \alpha$. Notice, that if $\sup(M\cap\alpha) = \alpha$ then α is a minimum of a block in N and consequently $\alpha \in \mathcal{S}_r \subset \mathcal{D}_r$ by (9) in r, which is also the case if $\alpha = \sup(M\cap N)$.

However, if $\alpha < \sup(M\cap N)$ then $M\cap N \in \mathscr{N}$. Therefore, by compatibility between α and $M\cap N$ in s, $\min(M\setminus\alpha) = \min((M\cap N)\setminus\alpha) \in \mathcal{S}_s \subset \mathcal{S}_t$ and $\sup(M\cap\alpha) = \sup((M\cap N)\cap\alpha) \in \mathcal{D}_s \subset \mathcal{D}_t$. Also, if there is no $\delta \in \mathcal{D}_t$ such that $C_\alpha \in \mathrm{Lim}(C_\delta)$ then $C_\alpha \cap \sup(M\cap\alpha) = C_\alpha \cap \sup((M\cap N)\cap\alpha)$ which is either an ω-sequence if $\sup(M\cap\alpha) = \alpha$, or finite if $\sup(M\cap\alpha) < \alpha$. If $\alpha \in \mathrm{Lim}(C_\delta)$ for some $\delta \in \mathcal{D}_s$ then $C_\alpha = C_\delta \cap \alpha$ by (6) in s. If such δ is in $\mathcal{D}_r \setminus \mathcal{D}_s$ then, by (4) which was already proved for t, $C_\alpha = C_\delta \cap \alpha$.

If $\alpha \in M$ then we have to consider two possibilities. The first case is if $M\cap N \in \mathscr{N}$. Then, by (6a) in s, $C_\alpha \in \mathscr{M}$. Also, if $\sup(M\cap\alpha) < \alpha$ then $\sup(M\cap\alpha) = \sup((M\cap N)\cap\alpha) \in \mathcal{D}_s \subset \mathcal{D}_t$, and $\min(M\setminus\alpha) = \min((M\cap N)\setminus\alpha) \in \mathcal{S}_s \subset \mathcal{S}_t$, both by (6b) in s.

If, on the other hand, $M\cap N$ is an initial segment of N, and if $\mathrm{cf}(\alpha) = \omega$ then, since $C_\alpha \in [\sup(M\cap N)]^\omega \cap \mathscr{N}$, by Definition 4.1, $C_\alpha \in \mathscr{M}$. If $\mathrm{cf}(\alpha) = \omega_1$ then $C_\alpha = C\setminus\beta$ for some $C \in \mathscr{E}_\alpha \in \mathscr{M}$ and $\beta \in \mathcal{D}_s \cap \alpha \subset \mathscr{N}$. Since $\beta < \alpha$ and $M\cap N$ is an initial segment of N, we have $\beta \in \mathscr{M}$, hence $C_\alpha \in \mathscr{M}$. Note that if $\mathcal{D}_s \cap \alpha = \emptyset$ then C_α is in $\mathscr{E}_\alpha$, and by extension in $\mathscr{M}$, by itself.

We also have to consider the possibility that $\sup(M\cap\alpha) < \alpha$. Then $\sup(M\cap\alpha) \geq \sup(N\cap\alpha)$. Here we are in trouble because α knows nothing about M, so there is no reason to believe that $\sup(M\cap\alpha)$ is either in $\mathcal{D}_r$ or in $\mathcal{D}_s$. In fact, it is certainly not in $\mathcal{D}_r$ because then α would also be in $\mathcal{D}_r$ by (6b) in r applied on M and $\sup(M\cap\alpha)$. This is the reason why we have to enlarge $\mathcal{D}_t$ with every possible $\beta := \sup(M\cap\alpha)$ for all pairs $\alpha \in \mathcal{D}_s \setminus \mathcal{D}_r$ and countable $M \in \mathcal{M}_r \setminus \mathcal{M}_s$ such that $M\cap N \notin \mathscr{N}$, $\alpha < \sup(M\cap N)$ and $\sup(M\cap\alpha) < \alpha$. Notice, that this way we also prove the consistency between α and N. C_β's have to be chosen according to (6d) in t. The precise definition will follow. Meanwhile, it is worth mentioning that $\beta \notin \mathcal{D}_t$, hence C_β is a new club, and we are not in danger of overwriting an existing club with a new and different one. If β were present in t it could only come from r, as $\beta \notin N$. But then α itself would be present by (6b) in r. Also, $\mathrm{cf}(\beta) = \omega$, hence we need not worry about $\mathscr{E}_\beta$.

We have to check that every new β and its club are compatible with everything in t. Club C_β is compatible with all the other clubs as well as with $\mathcal{S}_t$ because of its construction. Point β is compatible with $\mathcal{O}_t$ because α is – at least it will be, once we prove clause (7) in t. The only nontrivial thing to check is the compatibility between β and $M' \in \mathcal{M}_r \setminus \mathcal{M}_s$. If $M' \in \mathcal{M}_s$ then the compatibility follows from compatibility between α and M' in s. Similarly, we only have to consider M' such that $M'\cap N \notin \mathscr{N}$, otherwise we simply use either the compatibility between $M'\cap N$ and α in s if $\alpha < \sup(M'\cap N)$, or the compatibility between M' and N in r if $\alpha \geq \sup(M'\cap N)$.

Suppose that M' is as described. If $\sup(M\cap M')<\beta<\sup(M')$ then the block of M with supremum β lies in a gap of M'. But then, by (9) in r, $\min(M'\setminus\beta)\in\mathcal{S}_r$, and consequently $\sup(M'\cap\beta)\in\mathcal{D}_r$ by (6b) in r. In fact, in this case β was already present in r and did not have to be added. If $\beta=\sup(M\cap M')$ then $\beta\in\mathcal{D}_r$, hence $\alpha=\min(M\setminus\beta)\in\mathcal{S}_r$.

Suppose now that $\beta<\sup(M\cap M')$. If $M\cap M'\in\mathscr{M}'$ then $\beta\in M'$ and it is not even a minimum of a block. If $C_\beta=C_\alpha\cap\beta$ then $C_\beta\in\mathscr{M}'$, because $\alpha\in M'$. However, if C_β is an ω-sequence then it has to be chosen so that it is an element of every such $\mathscr{M}'$. Therefore, the precise definition of C_β is this: if $\beta\in C_\alpha$ then $C_\beta:=C_\alpha\cap\beta$, otherwise C_β is some ω-sequence cofinal in β such that $C_\beta\in\bigcap\{M'\in\mathcal{M}_r\setminus\mathcal{M}_s\mid\beta\in M'\}\prec H_{\omega_2}$. In the second case, we can find such ω-sequence by elementarity, since a finite intersection of elementary submodels of H_{ω_2} is itself an elementary submodel of H_{ω_2}.

If $M\cap M'\in\mathscr{M}$ and $\alpha\leq\sup(M'\cap N)$ then, because $M'\cap N$ is an initial segment of N, α is also a minimum of a block in M' and hence $\sup(M'\cap\beta)=\sup(M'\cap\alpha)$, which had to be added to $\mathcal{D}_t$ just like β. Additionally, β is inside a gap of M' and C_β is compatible with M' because it was constructed in the spirit of (6d). Now, if $\alpha>\sup(M'\cap N)$ then a block of N with α in it lies in a gap of M'. But then $\sup(M'\cap\beta)=\sup(M'\cap\alpha)\in\mathcal{D}_r\subset\mathcal{D}_t$ and $\min(M'\setminus\beta)=\min(M'\setminus\alpha)\in\mathcal{S}_r\subset\mathcal{S}_t$. Again, β is inside a gap of M' and C_β is compatible with M' because of its construction.

If $M\cap M'$ is an initial segment of both M and M' then they have the same gaps below $\sup(M\cap M')$ and β closes the gap in M' just as it does it in M. Also, C_β is then compatible with M' just as it is with M.

Clause (7): first consider arbitrary $(\beta',\beta]\in\mathcal{O}_r\setminus\mathcal{O}_s$ and $\alpha\in\mathcal{D}_s\setminus\mathcal{D}_r$. Since $(\beta',\beta]\cap N=\emptyset$ by (8) in r, and $\alpha\in N$, we can conclude that $\alpha\notin(\beta',\beta]$. Now, if $\alpha\in\mathcal{D}_r\setminus\mathcal{D}_s$ and $(\beta',\beta]\in\mathcal{O}_s\setminus\mathcal{O}_r$ then potentially $\alpha\in(\beta',\beta]$. In that case $\min(N\setminus\alpha)\in\mathcal{S}_r\cap\mathscr{N}\subset\mathcal{S}_s$ by (6b) in r. But $\min(N\setminus\alpha)\in(\beta',\beta]$ which is in a direct contradiction with (7) in s.

Clause (8): start with some $(\beta',\beta]\in\mathcal{O}_r\setminus\mathcal{O}_s$ and $M\in\mathcal{M}_s\setminus\mathcal{M}_r$. The interval is disjoint from N, hence it is disjoint from $M\subset N$.

Now consider arbitrary $M\in\mathcal{M}_r\setminus\mathcal{M}_s$ and $(\beta',\beta]\in\mathcal{O}_s\setminus\mathcal{O}_r$. Suppose that $(\beta',\beta]\cap M\neq\emptyset$ but $\{\beta',\beta\}\not\subset M$. If $\beta\geq\sup(M\cap N)$ then, by (9) in r, there is an x from the N-fence for M in the interval. But $x\in\mathcal{S}_r\cap\mathscr{N}\subset\mathcal{S}_s$, a contradiction. However, if $\beta<\sup(M\cap N)$ then, because $\{\beta',\beta\}\not\subset M$, $M\cap N$ is not an initial segment of N. This means that $M\cap N\in\mathscr{N}$. But then $(\beta',\beta]$ is inconsistent with $M\cap N$ in s, again a contradiction.

Clause (9): consider arbitrary models $M\in\mathcal{M}_r\setminus\mathcal{M}_s$ and $M'\in\mathcal{M}_s\setminus\mathcal{M}_r$. Let $\delta:=\sup(N\cap M)$, $\delta':=\sup(M'\cap M)=\sup(M'\cap N\cap M)\leq\delta$ and $\varepsilon:=\min(N\setminus\delta)$. Let us first establish the proper connection between their intersection and each of them.

First assume that $M\cap N\in\mathscr{N}$. Then $M\cap M'=M\cap N\cap M'$ is in the same correspondence with both $\mathscr{M}$ and $\mathscr{M}'$ as $(M\cap N)\cap M'$ is with $\mathscr{M}\cap\mathscr{N}$ and $\mathscr{M}'$ respectively. Notice, that $M\cap N$ is an initial segment of M. On the other hand, if $M\cap N$ is an initial segment of N then $M\cap M'$ is also an initial segment of $M'\subset N$. At the same time we can prove that $M\cap M'\in\mathscr{M}$, as long as we can prove that $M\cap M'\in[\delta]^\omega\cap\mathscr{N}$, because then we can envoke the fact that $[\delta]^\omega\cap\mathscr{N}\subset\mathscr{M}$. It is clear that $M\cap M'\in[\delta]^\omega$, since $\delta'\leq\delta$. If $M\cap M'=M'\cap\varepsilon\in\mathscr{N}$, then we are done.

To that end, first let $\lambda \in M \cap M'$. Then $\lambda \in M \cap N$, hence $\lambda < \delta \leq \varepsilon$. Therefore $M \cap M' \subset M' \cap \varepsilon$. For the other direction assume that $\lambda \in M' \cap \varepsilon$. Then $\lambda \in N \cap \varepsilon$, hence $\lambda \leq \delta$. If $\lambda = \delta$ then $\delta \in N$, which cannot happen because $M \cap N \notin \mathscr{N}$. Therefore $\lambda \in N \cap \delta \subset M$.

Let us now establish the existence of fences. The M-fence for M' is some subset of the union of the M-fence for N and (if $M \cap N \in \mathscr{N}$) the $M \cap N$-fence for M', because $\{\min(M \setminus \lambda) \mid \lambda \in M', \lambda > \sup(M \cap M')\} = \{\min(M \setminus \lambda) \mid \lambda \in N, \lambda > \sup(M \cap N)\} \cup \{\min((M \cap N) \setminus \lambda) \mid \lambda \in M', \sup(M \cap M') < \lambda \leq \sup(M \cap N)\}$, and this union is a subset of $\mathcal{S}_r \cup \mathcal{S}_s = \mathcal{S}_t$.

Let $x \subset \mathcal{S}_r \cap \mathscr{N}$ be the N-fence for M and $y \subset \mathcal{S}_s$ the M'-fence for $M \cap N$ (if $M \cap N \in \mathscr{N}$). Then the M'-fence for M is some subset of $\{\min(M' \setminus \lambda) \mid \lambda \in x \cup y \cup \{\min(N \setminus \sup(M \cap N))\}\} \subset \mathcal{S}_s$. $\surd$

5. Preservation of ω_2

We have thus far proved that forcing with P preserves ω_1. We also need ω_2 to be preserved. For that purpose we use a weaker version of closedness, which was also used in [11].

Definition 5.1. *Assume that forcing notion P preserves cardinals $< \delta$. P is δ-*presaturated *if for every $A \subset V$, $A \in V[G]$, with $|A|^{V[G]} < \delta$, there exists $A' \in V$ such that $|A'|^V < \delta$ and $A' \supset A$.*

Notice that since P preserves cardinals below δ, $|A'|^V = |A'|^{V[G]}$ as soon as $|A'|^V < \delta$. Hence we can omit the superscript.

Proposition 5.2. *Suppose δ is a regular cardinal in V. If P is δ-presaturated then P preserves δ.*

Proof. Suppose for contradiction that $f : \mu \to \delta$ is a bijection in $V[G]$ for some $\mu < \delta$. For $\alpha < \mu$, let $A'_\alpha \in V$ be such that $|A'_\alpha| < \delta$ and $f[\alpha + 1] \subset A'_\alpha$. Define a function $g : \mu \to \delta$ by $g(\alpha) := \sup(A'_\alpha \cap Ord) < \delta$ for $\alpha < \mu$. Then $g \in V$ and $f(\alpha) \leq g(\alpha)$ for all $\alpha < \mu$. Hence g is cofinal, and we get a contradiction. $\surd$

Lemma 5.3. *Let δ be a cardinal regular in V such that P preserves cardinals below δ. Suppose that for every collection $\mathcal{A}$ of fewer than δ antichains there exists a dense set $\mathscr{D} \subset P$ such that for every $p \in \mathscr{D}$, the set $\{q \in \bigcup \mathcal{A} \mid p$ and q are compatible$\}$ has size less than δ. Then P is δ-presaturated.*

Proof. Suppose $A \subset V$ and $|A|^{V[G]} < \delta$. Let $p \in G$ be a condition such that $p \Vdash$ "$|\underset{\sim}{A}| < \delta$". Therefore $p \Vdash$ "there exists $\mu < \delta$ such that $|\underset{\sim}{A}| = \mu$". Let $p_0 \geq p$, g and $\mu^* < \delta$ be such that $p_0 \Vdash$ "$\underset{\sim}{g} : \mu^* \to \underset{\sim}{A}$ is a bijection". For each $\alpha < \mu^*$ let $\mathcal{A}_\alpha$ be a maximal antichain of conditions in the set $\{q \mid (q \geq p_0 \wedge q$ decides $g(\alpha)) \vee q \perp p_0\}$. Hence $\mathcal{A}_\alpha$ is a maximal antichain above p_0.

Define $\mathcal{A} := \{\mathcal{A}_\alpha \mid \alpha < \mu^*\}$. Let $\mathscr{D}$ be a dense set guaranteed by the assumption, and let $p_1 \in \mathscr{D}$, $p_1 \geq p_0$. Then the set $X := \{q \in \bigcup_{\alpha < \mu^*} \mathcal{A}_\alpha \mid q$ is compatible with $p_1\}$ has size $< \delta$. Let $\Gamma := \{\beta \mid$ there exist $q \in X$ and $\alpha < \mu^*$ such that $q \Vdash$ "$\underset{\sim}{g}(\alpha) = \beta$"$\}$. Consider an arbitrary $\alpha < \mu^*$. Since $\mathcal{A}_\alpha$ is a maximal antichain there exists some $q \in \mathcal{A}_\alpha$, compatible with p_1, such that q decides $g(\alpha)$. Hence there exists β such that $q \Vdash$ "$\underset{\sim}{g}(\alpha) = \beta$", and therefore $\beta \in \Gamma$. Let r be a common upper bound for q and p_1. Then $r \Vdash$ "$\underset{\sim}{g}(\alpha) = \beta$", and since $r \geq p_0$, $p_0 \Vdash$ "there exists $\beta \in \Gamma$ such that $\underset{\sim}{g}(\alpha) = \beta$". It follows that $p_0 \Vdash$ "$\underset{\sim}{g}(\alpha) \in \Gamma$", so $p_0 \Vdash$ "$\underset{\sim}{g}[\mu^*] = \underset{\sim}{A} \subset \Gamma$".

Since δ is regular in V, we have $|\Gamma| < \delta$, therefore $p \Vdash$ "there exists $A' \in V$, $\underset{\sim}{A} \subset A'$ and $|A'| < \delta$". $\surd$

The next lemma shows that δ-presaturation is, in fact, a generalization of properness to cardinals above ω_1.

Lemma 5.4. *Let δ be a cardinal regular in V such that P preserves cardinals below δ. Suppose that θ is a large enough cardinal, and that for stationarily many models $\mathscr{N}$ of size $< \delta$ in $[H_\theta]^{<\delta}$ with $P \in \mathscr{N}$, and for each $p \in P \cap \mathscr{N}$, there exists an $\mathscr{N}$-generic extension $q \geq p$. Then P is δ-presaturated.*

Proof. Suppose $A \subset V$ and $\mu := |A|^{V[G]} < \delta$. Let f be a function and let $p \in G$ be a condition such that $p \Vdash$ "$\underset{\sim}{f} : \mu \to \underset{\sim}{A}$ is onto". Define $\mathfrak{N} := \{\mathscr{N} \prec H_\theta \mid |\mathscr{N}| < \delta, \{f, A, p, P\} \cup \mu \subset \mathscr{N}\}$. Consider some $\mathscr{N} \in \mathfrak{N}$. Let $q \geq p$ be a generic extension. Then for every $\xi < \mu$, the set $\mathscr{D}_\xi := \{r \in \mathscr{N} \mid r \text{ decides } f(\xi)\} \in \mathscr{N}$ is dense above q. Hence $q \Vdash$ "$\mathscr{D} \cap G \cap \mathscr{N} \neq \emptyset$". Therefore q forces that there exist $r_\xi \in G \cap \mathscr{N}$ and $x_\xi \in \mathscr{N}$ such that $r_\xi \Vdash$ "$\underset{\sim}{f}(\xi) = x_\xi$". It follows that $q \Vdash$ "$\underset{\sim}{A} \subset \mathscr{N}$", so $p \Vdash$ "there exists $A' \in V$, $\underset{\sim}{A} \subset A'$ and $|A'| < \delta$", A' being the model $\mathscr{N}$. $\surd$

Proposition 5.5. *P is ω_2-presaturated.*

Since presaturation is a generalization of properness, the proof will be very similar to the proof of properness. Actually, it will be slightly easier, because we will not work with arbitrary models of size ω_1 but only with such models that are in a way transitive below ω_2.

Definition 5.6. *Let $\theta > \omega_2$ be some cardinal. Define $\mathfrak{M}_2 := \{\mathscr{M} \prec H_\theta \mid |\mathscr{M}| = \omega_1, \omega_1 \cup \{\mathscr{E}\} \subset \mathscr{M}, [\mathscr{M}]^\omega \subset \mathscr{M}\}$.*

Recall that we have assumed CH so that the set $\mathfrak{M}_2$ is stationary in $[H_\theta]^{<\omega_2}$. If $\mathscr{M} \in \mathfrak{M}_2$ then $\mathscr{M} \cap \omega_2$ is some ordinal $\delta_M \in \omega_2$, since $\omega_1 \subset \mathscr{M}$ (see [8]). Additionally, if $A \in \mathscr{M}$ and $|A| \leq \omega_1$ then $A \subset \mathscr{M}$.

To prove Proposition 5.5, we first isolate a lemma which is an analogue of Lemma 4.10.

Lemma 5.7. *Let $\mathscr{N} \in \mathfrak{M}_2$, and $r \in P$ such that $\delta_N \in \mathcal{S}_r$. Then $r_{\mathscr{N}}$ defined by $\mathcal{F}_{r_{\mathscr{N}}} := \mathcal{F}_r \cap \mathscr{N}$, $\mathcal{S}_{r_{\mathscr{N}}} := (\mathcal{S}_r \cap \mathscr{N}) \cup \{\sup(M \cap N) \mid M \in \mathcal{M}_r \setminus \mathscr{N}\}$, $\mathcal{O}_{r_{\mathscr{N}}} := \mathcal{O}_r \cap \mathscr{N}$, $\mathcal{M}_{r_{\mathscr{N}}} := \{M \cap N \mid M \in \mathcal{M}_r\}$, is a condition in $P \cap \mathscr{N}$.*

Proof. Notice that, since $M \cap N$ is a countable subset of $\mathscr{N}$, it is in $\mathscr{N}$. Hence, $r_{\mathscr{N}} \in \mathscr{N}$. Also, if $M \in \mathscr{N}$ then $M = M \cap N \in \mathcal{M}_{r_{\mathscr{N}}}$. It is also of some importance that by clause (6a), $\mathcal{D}_{r_{\mathscr{N}}} = \mathcal{D}_r \cap \mathscr{N}$.

Now we have to show, just as in the proof of Lemma 4.10, that $r_{\mathscr{N}}$ is a condition. By Lemmas 3.2 and 3.3, $\mathscr{M} \cap \mathscr{N} \prec H_{\omega_2}$. At the same time, $\mathscr{M} \cap \mathscr{N} \in \mathfrak{M}_0$, therefore $M \cap N$ can be justifiably added to $\mathcal{M}_{r_{\mathscr{N}}}$. Since $\delta_N \in \mathcal{S}_r$, it follows that $\sup(M \cap N) \in \mathcal{D}_r \cap \mathscr{N} = \mathcal{D}_{r_{\mathscr{N}}}$ by clause (6b). We can safely put it into $\mathcal{S}_{r_{\mathscr{N}}}$ without violating clause (5), because we can apply Lemma 4.7.

Just as in the proof of Lemma 4.10, we only have to pay attention to clauses (6) and (9) in conjecture with a model of the form $M \cap N$.

If $\alpha \in M \cap N$ then $C_\alpha \in \mathscr{M}$ by (6a) in r. If $\mathrm{cf}(\alpha) = \omega$ then $C_\alpha \in \mathscr{N}$, because $\mathscr{N}$ is countably closed. If $\mathrm{cf}(\alpha) = \omega_1$ then $C_\alpha \in \mathscr{N}$, because $C_\alpha = C \setminus \beta$ for some $C \in \mathscr{E}_\alpha \subset \mathscr{N}$ and $\beta \in \mathcal{D}_r \cap \alpha \subset \mathscr{N}$. In any case, $C_\alpha \in \mathscr{M} \cap \mathscr{N}$. The danger with clauses (6c) and (6d) is if $\alpha \in \mathscr{N}$ is in a gap of $M \cap N$ and $\alpha \in \mathrm{Lim}(C_\gamma)$ for some $\gamma \in \mathcal{D}_r \setminus \mathscr{N}$. But then $\gamma > \delta_N \in \mathcal{S}_r$, hence $\alpha \notin \mathrm{Lim}(C_\gamma)$.

Let $M_1' := M_1 \cap N$ and $M_2' := M_2 \cap N$ be two models from $\mathcal{M}_{r_{\mathscr{N}}}$. Suppose that $M_1 \cap M_2$ is an initial segment of M_1 and $A \in [\lambda']^\omega \cap \mathscr{M}_1'$ where, $\lambda' := \sup(M_1' \cap M_2') \leq \sup(M_1 \cap M_2) =: \lambda$. Then $A \in [\lambda]^\omega \cap \mathscr{M}_1 \cap \mathscr{N} \subset \mathscr{M}_2 \cap \mathscr{N} = \mathscr{M}_2'$.

Suppose now that $M_1 \cap M_2 \in \mathscr{M}_1$. Then $\lambda' \in \mathscr{M}_1$, hence $M_1' \cap M_2' = M_1 \cap M_2 \cap \lambda' \in \mathscr{M}_1$. On the other hand, $M_1' \cap M_2'$ is a countable subset of $\mathscr{N}$, hence it is in $\mathscr{N}$. Therefore, $M_1' \cap M_2' \in \mathscr{M}_1 \cap \mathscr{N} = \mathscr{M}_1'$.

The M_1'-fence for M_2' is the M_1-fence for M_2 intersected with N. The M_2'-fence for M_1' is obtained in the same way. $\surd$

Possible complication: is C_δ in $\mathscr{E}_\delta$?

Proof (of Proposition 5.5). Let θ be a large enough cardinal. Pick $\mathscr{N} \in \mathfrak{M}_2$ and $p \in P \cap \mathscr{N}$. We extend p to q by putting δ_N into both $\mathcal{D}_p$ and $\mathcal{S}_p$. For the corresponding club C_{δ_N} we take $C \setminus \max(\mathcal{D}_p)$ for some $C \in \mathscr{E}_{\delta_N}$. Clearly, $q \in P$. We will prove that q is an $\mathscr{N}$-generic extension of p.

Suppose r is an arbitrary extension of q, and let $r_{\mathscr{N}}$ be as given by the previous lemma. For a fixed dense set $\mathscr{D} \subset P$, $\mathscr{D} \in \mathscr{N}$, extend $r_{\mathscr{N}}$ to $s \in \mathscr{D}$. Then $s \in \mathscr{N}$. As with properness, we will prove clause by clause of Definition 4.4 that $t := (\mathcal{F}_r \cup \mathcal{F}_s, \mathcal{S}_r \cup \mathcal{S}_s, \mathcal{O}_r \cup \mathcal{O}_s, \mathcal{M}_r \cup \mathcal{M}_s)$ is a condition. Fortunately, less effort will have to be invested, because $N \cap H_{\omega_2}$ has no gaps.

Clauses (1), (2) and (3) need no comments.

Clause (4): suppose that $\alpha \in \mathcal{D}_r \setminus \mathcal{D}_s$ and $\beta \in \mathcal{D}_s \setminus \mathcal{D}_r$. Then $C_\alpha \cap \mathscr{N}$ is a finite set because $\delta_N \in \mathcal{S}_r$, and $C_\beta \subset \mathscr{N}$, hence $\mathrm{Lim}(C_\alpha) \cap \mathrm{Lim}(C_\beta) = \emptyset$.

Clause (5): if $\alpha \in \mathcal{D}_r \setminus \mathcal{D}_s$ and $\sigma \in \mathcal{S}_s \setminus \mathcal{S}_r$ then $C_\alpha \cap \sigma \subset C_\alpha \cap \delta_N$, which is a finite set.

Clause (6): if $\alpha \in \mathcal{D}_r \setminus \mathcal{D}_s$ and $M \in \mathcal{M}_s \setminus \mathcal{M}_r$ then there is nothing to prove. Suppose now that $\alpha \in \mathcal{D}_s \setminus \mathcal{D}_r$ and $M \in \mathcal{M}_r \setminus \mathcal{M}_s$. Then $M \cap N \in \mathcal{M}_s$ and the compatibility between α and $M \cap N$ in s is transfered to the compatibility between α and M. The only potential problem would be, just as in the proof of the previous lemma, if $\alpha \geq \sup(M \cap N)$ and $\alpha \in \mathrm{Lim}(C_\gamma)$ for some $\gamma \in \mathcal{D}_r \setminus \mathcal{D}_s$. But as we saw, $\mathrm{Lim}(C_\gamma) \cap \mathscr{N} = \emptyset$, because $C_\gamma \cap \delta_N$ is a finite set.

Clause (7): if $\alpha \in \mathcal{D}_r \setminus \mathcal{D}_s$ and $(\beta', \beta] \in \mathcal{O}_s \setminus \mathcal{O}_r$ then $(\beta', \beta] \subset \mathscr{N}$, hence $\alpha \notin (\beta', \beta]$. Suppose now that $\alpha \in \mathcal{D}_s \setminus \mathcal{D}_r$ and $(\beta', \beta] \in \mathcal{O}_r \setminus \mathcal{O}_s$. Since $\delta_N \in \mathcal{S}_r$, we have $(\beta', \beta] \cap \mathscr{N} = \emptyset$, hence $\alpha \notin (\beta', \beta]$.

Clause (8): if $M \in \mathcal{M}_s \setminus \mathcal{M}_r$ and $(\beta', \beta] \in \mathcal{O}_r \setminus \mathcal{O}_s$ then $(\beta', \beta] \cap \mathscr{M} = \emptyset$. Consider now some $M \in \mathcal{M}_r \setminus \mathcal{M}_s$ and $(\beta', \beta] \in \mathcal{O}_s \setminus \mathcal{O}_r$. Then $(\beta', \beta]$ and $M \cap N$ are compatible in s. If $(\beta', \beta] \in \mathscr{M} \cap \mathscr{N}$ then $(\beta', \beta] \in \mathscr{M}$. If $(\beta', \beta] \cap (\mathscr{M} \cap \mathscr{N}) = \emptyset$ then $(\beta', \beta] \cap \mathscr{M} = ((\beta', \beta] \cap \mathscr{N})) \cap \mathscr{M} = \emptyset$.

Clause (9): consider two models $M \in \mathcal{M}_r \setminus \mathcal{M}_s$ and $M' \in \mathcal{M}_s \setminus \mathcal{M}_r$. Then $M \cap N$ and M' are compatible in s. If $M \cap N \cap M' \in \mathscr{M} \cap \mathscr{N}$ then $M \cap M' = M \cap M' \cap N \in \mathscr{M}$. Now suppose that $M \cap N \cap M'$ is an initial segment of $M \cap N$ and let $A \in [\lambda]^\omega \cap \mathscr{M}$, where $\lambda := \sup(M \cap M') = \sup(M \cap N \cap M') \in N$. Then $A \in \mathscr{N}$, since $\mathscr{N}$ is countably closed. Hence $A \in [\lambda]^\omega \cap \mathscr{M} \cap \mathscr{N} \subset \mathscr{M}'$.

On the other hand, if $M \cap N \cap M' \in \mathscr{M}'$ then $M \cap M' = M \cap M' \cap N \in \mathscr{M}'$. However, if $M \cap N \cap M'$ is an initial segment of M' and $A \in [\lambda]^\omega \cap \mathscr{M}' \subset \mathscr{M} \cap \mathscr{N}$, then $A \in \mathscr{M}$, hence $M \cap M'$ is an initial segment of M'. $\surd$

Corollary 5.8. *Forcing P preserves cardinals.*

Proof. P has the $\boldsymbol{\omega_3}$**-c.c.** because, assuming $2^{\omega_1} = \omega_2$, $|P| = \omega_2$. Hence it preserves cardinals $\geq \omega_3$. It preserves ω_2 because it is ω_2-presaturated. And it preserves ω_1 because it is proper. $\surd$

Definition 5.9. *Let $G \subset P$ be a generic set. Define $\mathcal{F} := \bigcup_{p\in G} \mathcal{F}_p$, and $\mathcal{C} := \mathrm{dom}(\mathcal{F})$.*

We have no reason to believe that $\mathcal{C} = \mathrm{Lim}(\omega_2)$. The usual density argument fails in this case. Namely, we cannot extend a given $p \in P$ with $\alpha \notin \mathcal{D}_p$ to $q \in P$ such that $\alpha \in \mathcal{D}_q$. At least not for any $p \in P$ and $\alpha < \omega_2$. Suppose $p = (\{(\omega_1 + \omega, C)\}, \emptyset, \emptyset, \{M\})$ and $C = \omega + 1 \cup [\omega_1, \omega_1 + \omega) \in \mathscr{M}$. If we want to extend $\mathcal{D}_p$ to $\alpha \in [\delta_M, \omega_1]$ then we have to add ω_1 to $\mathcal{S}_p$ for clause (6b) to hold. But then ω_1 and C violate clause (5).

Proposition 5.10. *$\mathcal{C}$ is unbounded in ω_2.*

Proof. Define $\mathscr{D}_\alpha := \{p \in P \mid \max(\mathcal{D}_p) > \alpha\}$ for $\alpha < \omega_2$. Consider arbitrary $p \in P$ and assume that $p \notin \mathscr{D}_\alpha$. Now let $\alpha' := \sup(\mathcal{D}_p \cup \bigcup \mathcal{O}_p \cup \bigcup \mathcal{M}_p) < \omega_2$ and $q := (\mathcal{F}_p \cup \{\alpha' + \omega, (\alpha', \alpha' + \omega)\}, \mathcal{S}_p, \mathcal{O}_p, \mathcal{M}_p)$. Clearly, $q \in P$, $q \geq p$ and $q \in \mathscr{D}_\alpha$, hence $\mathscr{D}_\alpha$ is dense in P for every $\alpha < \omega_2$. It follows that $\mathcal{C}$ is unbounded in ω_2. $\surd$

To prove that $\mathcal{C}$ is closed, we need the following lemma.

Lemma 5.11. *Suppose that $p \in P$ is such that $\alpha \notin \mathcal{D}_p$. There exists an extension $q \geq p$ such that there is some interval $(\beta', \beta] \in \mathcal{O}_q$ with $\alpha \in (\beta', \beta]$.*

Proof. Assume that $\alpha \notin (\beta', \beta]$ for every $(\beta', \beta] \in \mathcal{O}_p$. Define $A := \{M \in \mathcal{M}_p \mid \alpha \notin M,\ \sup(M \cap \alpha) = \alpha\}$ and $B := \{M \in \mathcal{M}_p \mid \alpha \in M\}$. Notice that by clause (3) of Definition 4.4, $\alpha < \sup(M)$ for every $M \in A$.

First assume that $B = \emptyset$. If $A = \emptyset$ then define $\beta' := \sup((\mathcal{D}_p \cup \bigcup \mathcal{M}_p) \cap \alpha) < \alpha$ and $\beta := \alpha$. Obviously $(\beta', \beta] \cap \mathcal{D}_p = \emptyset$ and $(\beta', \beta] \cap M = \emptyset$ for every $M \in \mathcal{M}_p$.

If $A \neq \emptyset$ then let β' be any ordinal in $\bigcap A$ strictly between $\sup((\mathcal{D}_p \cup \bigcup(\mathcal{M}_p \setminus A)) \cap \alpha)$ and α. Such an ordinal exists because α is a supremum of some block in every $M \in A$. Namely, if $M, M' \in A$ have interchanging blocks with supremum α then there has to be infinitely many such blocks, but that cannot happen neither below $\sup(M \cap M')$ nor above it (see Remark 4.3). Therefore only one model from A can have α as a supremum of some sequence of blocks rather than a supremum of a certain block. To be precise, there can be more such models but the sequence must be the same from some point on. In this case we can still find β' as described.

On the other hand, if $M, M' \in A$ then $\min(M \setminus \alpha) = \min(M' \setminus \alpha)$. That is true because if $\alpha = \sup(M \cap M')$ then $\alpha \in \mathcal{D}_p$. But if $\alpha < \sup(M \cap M')$ and if, for instance, $\alpha' := \min(M \setminus \alpha) < \min(M' \setminus \alpha)$ then $M \cap M'$ is not an initial segment of M, hence it must be an element of M, which means that $\alpha = \sup((M \cap M')) \cap \alpha') \in M$. However, that contradicts the original assumption that $\alpha \notin M$. Define $\beta := \min(M \setminus \alpha)$ for some (i.e. every) $M \in A$. Interval $(\beta', \beta]$ is an element of every $M \in A$. If $\gamma \in (\beta', \beta]$ for some $\gamma \in \mathcal{D}_p$ then $\alpha \in \mathcal{D}_p$ by clause (6b) of Definition 4.4. If $(\beta', \beta] \cap M' \neq \emptyset$ for some $M' \in \mathcal{M}_p \setminus A$ then we have to consider two distinct possibilities. If it happens above the $\sup(M \cap M')$ for some $M \in A$ then a block of M' can only lie inside a gap of M, which means that β is in the M-fence for M', and, again by (6b), $\alpha \in \mathcal{D}_p$.

If it occurs below the $\sup(M \cap M')$ for some $M \in A$ then it can only happen if $(\beta', \beta] \cap M' = \{\beta\}$. In this case $(\beta', \beta]$ and M' are not compatible. Let $A' := \{M' \in \mathcal{M}_p \setminus A \mid \beta \in M'\}$. We have to extend the interval $(\beta', \beta]$ downwards so that $\beta' \in M'$ for every $M' \in A'$ but $\beta' > \sup((\mathcal{D}_p \cup \bigcup(\mathcal{M}_p \setminus (A \cup A'))) \cap \alpha)$. If we can find such β' then this new interval will still be compatible with every $M \in \mathcal{M}_p \setminus A'$, and it will obviously be compatible with every $M' \in A'$. We can be sure that there is no $\gamma \in \mathcal{D}_p$

in the gap between $\sup(M' \cap \beta)$ and β, because otherwise $\beta \in \mathcal{D}_p$, hence $\alpha \in \mathcal{D}_p$. So this takes care of $\mathcal{D}_p$. Can we find β' so that the new interval does not intersect any $M'' \in \mathcal{M}_p \setminus (A \cup A')$? Suppose M'' is the reason that the answer is negative. Then $\sup(M'' \cap \beta) \geq \sup(M' \cap \beta)$. If $\sup(M'' \cap \beta) \geq \sup(M' \cap M'')$ then β is in the M'-fence for M'' and we derive a contradiction. If $\sup(M'' \cap \beta) < \sup(M' \cap M'')$ then either we get a contradiction by Remark 4.3 if $\sup(M'' \cap \beta) > \sup(M' \cap \beta)$, or, if $\sup(M'' \cap \beta) = \sup(M' \cap \beta)$, we can conclude that $M' \cap M'' \in M'$ (because $\beta \in M' \setminus M''$), hence $\sup(M' \cap \beta) = \sup((M' \cap M'') \cap \beta) \in M'$, again a contradiction. We should check that we can find β' so that it lies in every $M \in A$. If $M \in A$ is a counterexample then the whole block of M lies inside of a gap in some $M' \in A'$. It happens below the $\sup(M \cap M')$, as $\beta \in M \cap M'$, hence we can use Remark 4.3 to get a contradiction.

Now assume that $B \neq \emptyset$ and $A = \emptyset$. If $\sup(M \cap \alpha) = \alpha$ for every $M \in B$ then, similarly as above, let β' be any ordinal in $\bigcap B$ strictly between $\sup((\mathcal{D}_p \cup \bigcup(\mathcal{M}_p \setminus B)) \cap \alpha)$ and α, and $\beta := \alpha$. Then $(\beta', \beta] \in M$ for every $M \in B$ while it is disjoint from $\mathcal{D}_p$ and every $M' \in \mathcal{M}_p \setminus B$. If there is some $M \in B$ such that $\sup(M \cap \alpha) < \alpha$ then we construct $(\beta', \beta]$ in the very same way, but we have to check that $\sup((\mathcal{D}_p \cup \bigcup(\mathcal{M}_p \setminus B)) \cap \alpha) < \sup(M \cap \alpha)$ so that β' can be taken from M. If there is some $\gamma \in \mathcal{D}_p$ between $\sup(M \cap \alpha)$ and α then $\alpha \in \mathcal{D}_p$. The same is true if there is a block of some $M' \in \mathcal{M}_p \setminus B$ inside the gap of M just below α and $\sup(M \cap \alpha) \geq \sup(M \cap M')$. However, if $\sup(M \cap \alpha) < \sup(M \cap M')$ then there can be nothing from M' inside the gap of M below α, because, as several times before, that would make it impossible for $M \cap M'$ to be an initial segment of either M or M'. This follows from the fact that $\alpha \in M \setminus M'$.

Finally assume that $B \neq \emptyset$ and $A \neq \emptyset$. If $\sup(M \cap \alpha) = \alpha$ for every $M \in B$ then let β' be any ordinal in $\bigcap(B \cup A)$ strictly between $\sup((\mathcal{D}_p \cup \bigcup(\mathcal{M}_p \setminus (B \cup A))) \cap \alpha)$ and α. To define β, first recall that if $M, M' \in A$ then $\min(M \setminus \alpha) = \min(M' \setminus \alpha)$. Additionally, if $M \in B$ and $M' \in A$ then $\min(M' \setminus \alpha) \in M$. This is true because $\alpha < \sup(M \cap M')$, hence if it were not true then $M \cap M'$ would not be an initial segment of either M or M'. Notice here that if $\alpha = \sup(M \cap M')$ then $\alpha \in \mathcal{D}_p$. Define $\beta := \min(M' \setminus \alpha)$ for some $M' \in A$. Then $(\beta', \beta] \in M$ for every $M \in B \cup A$. It is also disjoint from $\mathcal{D}_p$ and every $M' \in \mathcal{M}_p \setminus (B \cup A)$, because otherwise $\alpha \in \mathcal{D}_p$. The only possible exception is if $(\beta', \beta] \cap M' = \{\beta\}$ for some $M' \in \mathcal{M}_p \setminus (B \cup A)$. This situation is dealt with in the same manner as in the case if $B = \emptyset$ and $A \neq \emptyset$. If $\sup(M \cap \alpha) < \alpha$ for some $M \in B$ while at the same time $\sup(M' \cap \alpha) = \alpha$ for some $M' \in A$ then it has cofinality ω_1 (see Lemma 3.4) and ω.

In all the cases we were able to find an interval $(\beta', \beta]$ compatible with everything in p, such that $\alpha \in (\beta', \beta]$. Hence, $q := (\mathcal{F}_p, \mathcal{S}_p, \mathcal{O}_p \cup \{(\beta', \beta]\}, \mathcal{M}_p)$ is the desired extension of p. $\surd$

Proposition 5.12. *$\mathcal{C}$ is closed in ω_2.*

Proof. Suppose for contradiction that $p \in G$ is such that $p \Vdash$ "$\alpha \in \mathrm{Lim}(\mathcal{C})$ but $\alpha \notin \mathcal{C}$" for some $\alpha < \omega_2$. Then $\alpha \notin \mathcal{D}_p$. Let q be the extension given by previous lemma. But then $q \Vdash$ "$\alpha \notin \mathrm{Lim}(\mathcal{C})$", which contradicts the fact that $p \Vdash$ "$\alpha \in \mathrm{Lim}(\mathcal{C})$". $\surd$

What we have created might not be a $\square_{\omega_1}$ sequence, but the next lemma shows that we can now extend our square-like sequence to the whole $\mathrm{Lim}(\omega_2)$.

Lemma 5.13. *Let κ be a regular cardinal $> \omega$. Suppose that $\mathcal{C} \subset \mathrm{Lim}(\kappa^+)$ is a club of κ^+ and $\langle C_\alpha \mid \alpha \in \mathcal{C}\rangle$ is a square-like sequence. Then there exists a square sequence on κ^+.*

Proof. The idea is to throw away every ordinal which is not in $\mathcal{C}$, effectively making $\mathcal{C}$ equal to κ^+. In fact, keeping only limit points of $\mathcal{C}$ will suffice. Thus, let $\mathcal{E} := \mathrm{Lim}(\mathcal{C}) \setminus \{\kappa^+\}$. $\mathcal{E}$ is stil a club of κ^+. For every $\alpha \in \mathrm{Lim}(\mathcal{E}) \setminus \{\kappa^+\}$ define $D_\alpha := C_\alpha \cap \mathcal{E}$. Since $\mathcal{E} \cap \alpha$ is a club in α for every $\alpha \in \mathrm{Lim}(\mathcal{E}) \setminus \{\kappa^+\}$, D_α is a club in α. Suppose that $\beta \in \mathrm{Lim}(D_\alpha)$ for some $\beta < \alpha$. Then β is a limit point of both $\mathcal{E}$ and C_α, and $D_\beta = C_\beta \cap \mathcal{E} = C_\alpha \cap \beta \cap \mathcal{E} = D_\alpha \cap \beta$. Also, if $\mathrm{cf}(\alpha) < \kappa$ then $|D_\alpha| < \kappa$. Hence, $\langle D_\alpha \mid \alpha \in \mathrm{Lim}(\mathcal{E}) \setminus \{\kappa^+\}\rangle$ is a square-like sequence.

Let $\{\gamma_i \mid i < \kappa^+\}$ be an increasing enumeration of $\mathcal{E}$. For $i \in \mathrm{Lim}(\kappa^+)$ define $E_i := \{j < i \mid \gamma_j \in D_{\gamma_i}\} = \gamma^{-1}[D_{\gamma_i}]$. It is a club in i because γ is a continuous function. Let us prove that $\langle E_i \mid i \in \mathrm{Lim}(\kappa^+)\rangle$ is a square sequence. If $i < j$ and $i \in \mathrm{Lim}(E_j)$ then $\gamma_i \in \mathrm{Lim}(D_{\gamma_j})$. Hence, $D_{\gamma_i} = D_{\gamma_j} \cap \gamma_i$. Therefore, $E_i = \gamma^{-1}[D_{\gamma_i}] = \gamma^{-1}[D_{\gamma_j} \cap \gamma_i] = \gamma^{-1}[D_{\gamma_j}] \cap i = E_j \cap i$. If $\mathrm{cf}(i) < \kappa$ then $\mathrm{cf}(\gamma_i) < \kappa$, hence $|E_i| = |D_{\gamma_i}| < \kappa$. $\surd$

Corollary 5.14. $V[G] \models \square_{\omega_1}$.

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